Perturbation-induced dynamics of dark solitons

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We study analytically and numerically the effect of perturbations on (spatial and temporal) dark optical solitons. Our purpose is to elaborate a general analytical approach to describe the dynamics of dark solitons in the presence of physically important effects which break integrability of the primary nonlinear Schrödinger equation. We show that the corresponding perturbation theory differs for the cases of constant and varying backgrounds which support the dark solitons. We present a general formalism describing the perturbation-induced dynamics for both cases and also analyze the influence of several physically important effects, such as linear and two-photon absorption, Raman self-induced scattering, gain with saturation, on the propagation of the dark soliton. As we show, the perturbation-induced dynamics of a dark soliton may be treated as a result of the combined effect of the background evolution and internal soliton dynamics, the latter being characterized by the soliton phase angle. A similar approach is applied to the problem of the dark-soliton propagation on a finite-width background. We analyze adiabatic modification of a dark pulse propagating on a dispersively spreading finite-width background, and we prove analytically that a frequency chirp of the background does not affect the soliton motion. As a matter of fact, the results obtained describe the perturbation-induced dynamics of dark solitons in the so-called adiabatic approximation and, as we show for all the cases analyzed, they are in excellent agreement with direct numerical simulations of the corresponding perturbed nonlinear Schrödinger equation, provided the effects produced by the emitted radiation are small.

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I. INTRODUCTION

Light solitons in time (temporal solitons) and space (spatial solitons) have been the object of intensive theoretical and experimental studies during the last three decades. The solitons, localized-in-time optical pulses or bounded-in-space optical beams, evolve from a nonlinear change in the refractive index of the material, induced by the light-intensity distribution. When the combined effects of the refractive-index nonlinearity and the pulse dispersion (in the case of temporal solitons) or diffraction (in the case of spatial solitons) exactly compensate each other, the pulse (or beam) propagates without change of its shape, being self-trapped by the waveguide nonlinearity. The nonlinear effects which are responsible for the soliton formation are, in general, Kerr-like effects, inducing local index changes proportional to the local light power. In this case, the main nonlinear equation governing the pulse evolution is the famous nonlinear Schrödinger (NLS) equation,

$$i\frac{\partial u}{\partial z}+\frac{\sigma}{2}\frac{\partial^2 u}{\partial \xi^2}\pm|u|^2u=0, \tag{1}$$

where u is the (complex) amplitude envelope of the electric field, z is the propagation distance along the optical waveguide or fiber, and the variable ξ has a different sense for temporal or spatial problems. In the case of temporal solitons observed in optical fibers (see, e.g., Refs. [1, 2] and references therein) the variable ξ stands for a retarded time measured in a frame of reference moving along the fiber at the group velocity. In this case the Kerr effect always produces a self-focusing contribution to the refractive index of the silica [the sign "+" in Eq. (1)] but the group-velocity dispersion (GVD) is known to vanish at a wavelength of about 1.3 μm so that it is positive ($\sigma=-1$) at larger wavelengths and negative ($\sigma=+1$) at shorter ones. In the case of spatial optical solitons (see, e.g., Ref. [3] and references therein) the variable ξ stands for the transverse coordinate, so that σ is always positive but nonlinearity itself may change sign, being positive [the sign "+" in Eq. (1)] for the so-called self-focusing nonlinear media, or negative [the sign "−" in Eq. (1)] for defocusing ones.

In the case of the anomalous GVD in fibers or the self-focusing nonlinearity in planar waveguides, the continuous-wave (cw) solution of the NLS equation (1) becomes modulationally unstable and it breaks into a chain of localized pulses, the so-called *bright solitons.* Soliton propagation of bright optical pulses has been verified in a number of elegant experiments (see, e.g., a work by Mollenauer *et al.,* [4] as well as more recent investigations of the soliton transmission in fibers [5]).

In the case of the normal GVD in fibers or the self-defocusing nonlinearity in waveguides, there are no bright solitons; instead pulses (or beams) undergo enhanced dispersive (diffractive) broadening and chirping. However, in this case the cw solution is modulationally stable, and the soliton pulses appear as localized nonlinear excitations of a background wave. The interest to analyze the dark-soliton propagation in optical models has been initiated by several experimental observations of temporal dark solitons in optical fibers [6–8] and spatial dark solitons in laser beams and planar waveguides [9–11]. We would like to note that such an experimental success to observe dark solitons may be mostly explained by the possibility to produce light beams of a high intensity or pulses with a duration of several picoseconds, but the physics underlying the dark-soliton propagation of the electromagnetic wave envelope is a rather fundamental phenomenon. To support this statement, we mention recent experiments [12] which reported the observation of dark solitons in thin magnetic films.

Properties of dark solitons have been described in many theoretical and several experimental papers (see, e.g., Refs. [13]–[35]) and the present state-of-the-art results in this field have been summarized in two review papers [36, 37]. As has been recently proposed in [29], various types of optical switching devices may be based on the propagation and interaction of dark spatial solitons which, similar to bright spatial solitons [38], may guide a probe optical beam. Such devices have very interesting properties, e.g., they may conserve the soliton steering angle, the key characteristic of the spatial soliton switching, even in the presence of the two-photon absorption [39], the effect which may have a dramatic influence on bright spatial solitons [40].

In many physical applications, the solitons are affected by different (even small) perturbations which may drastically change the pulse propagation. The perturbation theory for solitons is a rather well elaborated subject of the nonlinear dynamics of the soliton-bearing systems (see, e.g., the review paper [41]); however, up to now such a theory mostly covered the case of bright solitons, not proposing an analogous approach for dark solitons. Several analytical works have treated the perturbation-induced dynamics of dark solitons (see, e.g., some examples in Refs. [17–20, 23, 24, 33] and some other cases in [37]), but up to now there was no systematic approach to analyze the effect of small perturbations on the propagation of dark solitons. The main problem which does not allow us to apply immediately a general method of perturbation theory for solitons [41] is the background which supports the dark-soliton propagation: in many physically important cases perturbation leads to an induced dynamics of the background so that the perturbation theory cannot be straightforwardly applied to that case.

The present paper proposes a systematic way for studying the effects of small perturbations on dark solitons in the framework of the so-called adiabatic approximation. We show that the corresponding approach must differ for the cases of constant and varying background. For both the cases we show how to derive effective equations for the soliton parameters, and we point out that the perturbation-induced dynamics of dark solitons may be understood as a combined effect of the background evolution and the "internal" soliton motion described by the soliton phase angle. We apply our general approach to several physically important perturbations which appear in the physics of short-pulse propagation in optical fibers and self-trapped beams in planar waveguides, such as linear or two-photon (nonlinear) absorption, Raman self-scattering, gain with saturation, and we confirm our analytical predictions based on the adiabatic approximation by direct numerical simulations with rather good agreement.

The paper is organized as follows. In Sec. II we discuss the integrals of motion for the NLS equation showing how to describe properly the background wave and localized (dark) pulses which are propagating on it. In this consideration we naturally come to the idea of renormalized integrals of motion to dark solitons as recently introduced in Ref. [33] but which has been known, as a matter of fact, much earlier since the work of Lieb on nonlinear excitations of an interacting Bose gas [42] (see also Ref. [43] for review). Section III is devoted to the basic approach of the perturbation theory for dark solitons. As we show, the method does differ for the cases of constant and varying background so that it allows us also to analyze the cases when the background is changing in the presence of perturbations. In Sec. IV we consider several physically important cases when the general adiabatic approach may be applied to treat the effect of perturbations on the dynamics of a dark soliton. In particular, we consider two-photon and linear absorption, the Raman self-scattering effect, and linear gain with saturation. In all the cases mentioned above we compare the results of the adiabatic approximation with the corresponding results of direct numerical simulations of the primary NLS equation, and we observe an excellent agreement provided radiation effects are small. Despite the fact that the problem of dark solitons on a finite-width background is not perturbative, in Sec. V we show that it may be successfully analyzed in the framework of the approach developed in the present paper. In particular, we show analytically *why* in some special cases dark solitons on a finite-width background behave similarly to those on a constant background, even being not exact solitons. In Sec. VI we summarize our results and discuss other possible applications of the perturbation theory for dark solitons developed in this paper.

II. INTEGRALS OF MOTION FOR DARK SOLITONS

Most of the applications we deal with in Sec. IV are related to spatial dark solitons in nonlinear optical wave guides, so that to describe general properties of dark solitons, we take the NLS equation in the form

$$i\frac{\partial u}{\partial z} + \frac{1}{2}\frac{\partial^2 u}{\partial x^2} - |u|^2 u = 0. \tag{2}$$

As has been mentioned above, Eq. (2) describes the stationary beam propagation in a self-defocusing Kerr-type nonlinear optical medium, and z and x are the propagation distances along the waveguide and transverse coordinate, respectively. Equation (2) may be considered as the Euler equation which follows from the Lagrangian with the density

$$\mathcal{L} = \frac{i}{2}\left(u^*\frac{\partial u}{\partial z} - \frac{\partial u^*}{\partial z}u\right) - \frac{1}{2}\left|\frac{\partial u}{\partial x}\right|^2 - \frac{1}{2}|u|^4, \tag{3}$$

and it corresponds to the system Hamiltonian defined as the system energy

$$E = \int_{-\infty}^{+\infty} \mathcal{E}(x)dx, \quad \mathcal{E}(x) = \frac{1}{2}\left|\frac{\partial u}{\partial x}\right|^2 + \frac{1}{2}|u|^4. \tag{4}$$

As a matter of fact, Eq. (2) is exactly integrable [44] and it possesses an infinite quantity of integrals of motion. However, here we will be interested only in fundamental ones, which have a clear physical meaning. Because the system described by Eq. (2) is conservative, the total energy defined through Eq. (4) is conserved. Additionally, we will consider also the field momentum,

$$I = \int_{-\infty}^{+\infty} \mathcal{J}(x)dx, \quad \mathcal{J}(x) = \frac{i}{2}\left(u\frac{\partial u^*}{\partial x} - u^*\frac{\partial u}{\partial x}\right), \tag{5}$$

and the power

$$P = \int_{-\infty}^{+\infty} \mathcal{P}(x)dx, \quad \mathcal{P} = |u|^2. \tag{6}$$

The continuous-wave (cw) solution of Eq. (2) is given by the expression

$$u = u_0 e^{ikx - i\beta z}, \quad \beta = \frac{1}{2}k^2 + u_0^2. \tag{7}$$

It is easy to find that for the solution (7), the integrals of motion (4)–(6) take the following values:

$$P = u_0^2 L, \;\; I = k u_0^2 L = kP, \;\; E = \frac{P^2}{2L} + \frac{I^2}{2P}, \tag{8}$$

where L is the length of the system (or the width of the cw beam). As is well known, the cw solution (7) is *modulationally stable* in the defocusing medium. To show that property, let us consider stability of small variations around the exact solution (7),

$$u = (u_0 + v)e^{ikx - i\beta z + i\psi}, \tag{9}$$

where the function v and the gradients of the phase ψ are assumed to be small. Substituting Eq. (9) into the NLS equation (2), we come to a system of two coupled linear equations. Looking for solutions to the functions v and ψ in the standard form, $(v, \psi) \sim \exp(i\Omega z - iqx)$, we obtain the dispersion relation,

$$(\Omega - kq)^2 = q^2\left(u_0^2 + \frac{1}{4}q^2\right), \tag{10}$$

which shows that small excitations of the cw background are stable, and they are characterized by the minimum group velocity

$$c^2 = u_0^2. \tag{11}$$

Let us consider now a dark soliton as an excitation of the background wave (7). The solution of Eq. (2) describing a dark soliton with velocity V moving on a propagating background with the propagation constant β may be written in the most general form as

$$u(x,z) = u_0\left\{A\tanh[u_0 A(x - Vz)] + iB\right\}e^{ikx - i\beta z + i\phi_0}, \tag{12}$$

where, as above, the parameter $\beta = \frac{1}{2}k^2 + u_0^2$ characterizes the dispersion relation for the background wave, $\phi(0)$ is a constant phase, and the soliton and background parameters, A, B, and V, are connected by the relations

$$u_0 B = V - k, \qquad A^2 + B^2 = 1. \tag{13}$$

Therefore the dark-soliton solution (12) has three independent parameters, two of them are related to the background properties, u_0 and k, and only one characterizes the dark soliton itself. The asymptotics of the solution (12) coincide with those of the cw solution (7); however, the plane waves at the different edges, i.e., at $x \to \pm\infty$, are shifted in phase, and the total phase shift across the dark soliton is given by the result

$$\Delta\phi = 2\left[\tan^{-1}\left(\frac{B}{A}\right) - \frac{\pi}{2}\right] = -2\tan^{-1}\left(\frac{A}{B}\right). \tag{14}$$

Let us introduce now the integrals of motion characterizing the dark soliton itself. It is clear that the total integrals of motion of the NLS model describe the more complicated object "background plus soliton," so that we should modify the integrals (4)–(6) to extract the corresponding contributions of the background. After such a *renormalization*, the integrals of motion calculated for the solution (12) become finite. We will make this renormalization for the case $k = 0$, i.e., when the background is at rest and for the dark soliton we simply have $B = V/u_0 \equiv V/c$. It is clear that the power (6) must be modified as the following:

$$P_s = \int_{-\infty}^{\infty} dx(u_0^2 - |u|^2), \tag{15}$$

where the result (15) is obtained as a difference between the total power (6) and the corresponding value for the background, see Eq. (8). Calculating P_s for the solution (12) at $k = 0$, we find $P_s = 2u_0 A$. Renormalization of the field momentum is more complicated. To get a self-consistent description of properties of the dark soliton, one should extract from the total integral of motion (5) a contribution related to the phase difference (14) produced by the soliton. Contribution of the background into the field momentum (6) has the form $I = ku_0^2 L$ [see Eq.(8)], so that the soliton produces a similar contribution even for $k = 0$, because k differs from zero in the vicinity of the soliton, and the corresponding contribution for a varying $k = k(x)$ must be calculated as $u_0^2 \int k(x)dx = u_0^2 \Delta\phi$, where $k(x)$ is a function describing a local change of the background wave number in the vicinity of the soliton. As a result, the part of the field momentum related to the dark soliton itself is defined as

$$I_s = \frac{i}{2}\int_{-\infty}^{\infty} dx \left(u\frac{\partial u^*}{\partial x} - u^*\frac{\partial u}{\partial x}\right) - u_0^2 \Delta\phi, \tag{16}$$

where $\Delta\phi$ is given by Eq. (14). Substituting the solution (12) at $k = 0$ into Eq. (16), we find

$$I_s = -2V\sqrt{c^2 - V^2} + 2c^2 \tan^{-1}\left(\frac{\sqrt{c^2 - V^2}}{V}\right) \tag{17}$$

where c^2 is defined in Eq. (11).

At last, the soliton part of the total Hamiltonian (the system energy) may be defined as

$$E_s = \int_{-\infty}^{\infty} dx \left\{ \frac{1}{2}\left|\frac{\partial u}{\partial x}\right|^2 + \frac{1}{2}(|u|^2 - u_0^2)^2 \right\}, \tag{18}$$

and for the solution (12) it takes the form

$$E_s = \frac{4}{3}(c^2 - V^2)^{3/2}. \tag{19}$$

Differentiating the formulas (17) and (19) over the soliton velocity V, we find the simple relation

$$\frac{\partial E_s}{\partial I_s} = V, \tag{20}$$

which explicitly indicates that this renormalization of the integrals of motion leads to the standard relation of classical mechanics, so that a dark soliton, similar to a bright one, behaves like an effective particle.

The basic idea of the renormalization of the integrals of motion, which introduces the corresponding values for the dark soliton itself, does allow us to apply the perturbation theory for solitons considering the integrals of motion P, I, and E for the NLS equation excluding the corresponding contributions of the background.

III. PERTURBATION THEORY FOR DARK SOLITONS

A. Constant background

The analysis of the integrals of motion of the NLS equation to describe the so-called "soliton integrals of motion" introduced above allows us to apply a straightforward technique and to describe the perturbation-induced dynamics of dark solitons using the idea of the renormalization. First, we consider the case of the constant background when a perturbation does not change the parameters of the cw background.

Let us consider the perturbed NLS equation,

$$i\frac{\partial u}{\partial z} + \frac{1}{2}\frac{\partial^2 u}{\partial x^2} - |u|^2 u = \epsilon P(u), \tag{21}$$

where the term $\epsilon P(u)$ from the right-hand side stands for a small perturbation, ϵ being real. In this subsection we assume that such a perturbation does not change the cw background i.e., it vanishes at $|x| \to \infty$. Because the cw background wave $u = u_0 \exp(-iu_0^2 z)$ (we consider only the background at rest) does not change in the presence of the perturbation, the integrals of motion may be easily renormalized according to the procedure mentioned in Sec. II. To make such a renormalization more straightforward, let us introduce the new function $v(x,z)$ according to the relation

$$u(x,z) = u_0 e^{-iu_0^2 z} v(x,z), \tag{22}$$

and obtain the equation

$$i\frac{\partial v}{\partial \zeta} + \frac{1}{2}\frac{\partial^2 v}{\partial \xi^2} - (|v|^2 - 1)v = \epsilon \tilde{P}(v), \tag{23}$$

where $\epsilon\tilde{P}(v)$ is a renormalized perturbation $\epsilon P(u)$, $\zeta = u_0^2 z$, and $\xi = u_0 x$. Considering only the case of a nonpropagating background wave [i.e., $k = 0$ in Eq. (12)], we may now analyze how the parameters of the dark-soliton solution (12) will be changed due to the perturbation $\epsilon\tilde{P}(v)$ from the right-hand side of Eq. (23). At $k = 0$, the dark soliton (12) as a solution of Eq. (23) at $\epsilon = 0$ may be rewritten in the form

$$v(\zeta,\xi) = \cos\phi \tan Z - i\sin\phi, \quad Z = \eta(\xi - \Omega\zeta), \tag{24}$$

where $\eta = \cos\phi$ and $\Omega = \sin\phi$. Such a solution is characterized by the soliton phase angle ϕ ($|\phi| < \pi/2$) which describes the "darkness" of the soliton through the simple relation

$$|v|^2 = 1 - \frac{\cos^2\phi}{\cosh^2 Z}. \tag{25}$$

To treat analytically the influence of a perturbation $\epsilon\tilde{P}(v)$ on the parameters of the dark soliton (24), we use the so-called *adiabatic approximation* of the perturbation theory for solitons [41]. According to this approach, the parameters of the dark soliton (24) are considered as slowly varying in ζ (i.e. in z) but with the functional shape which remains unchanged, i.e. it is assumed to be described by Eq. (24), where we should modify Z as the following:

$$Z \to \cos\phi(\zeta)\left[\xi - \int d\zeta' \sin\phi(\zeta')\right]. \tag{26}$$

To derive the equation for the perturbation-induced evolution of the soliton phase $\phi(\zeta)$, we may use several (different but qualitatively similar) methods. In this paper we use the so-called Hamiltonian approach (see, e.g., Ref. [41] and references therein). To apply this method, we start from the Hamiltonian of the unperturbed system (23),

$$E = \int_{-\infty}^{\infty} d\xi \left\{ \frac{1}{2}\left|\frac{\partial v}{\partial \xi}\right|^2 + \frac{1}{2}\left(|v|^2 - 1\right)^2 \right\}, \tag{27}$$

which for the soliton solution (24) takes the value $E_s = \frac{4}{3}\cos^3\phi$ [cf. Eq. (19)]. Calculating the derivative of E over ζ and using Eq. (23), we find the result

$$\frac{dE}{d\zeta} = -\epsilon \int_{-\infty}^{\infty} d\xi \left[\tilde{P}(v)\frac{\partial v^*}{\partial \zeta} + \tilde{P}^*(v)\frac{\partial v}{\partial \zeta}\right]. \tag{28}$$

Assuming adiabatic change of the soliton parameters, in the lowest approximation we may use Eq. (28) to find the evolution of the soliton phase angle just using E_s for E. The resulting equation for $\phi(\zeta)$ may be written in the form

$$\frac{d\phi}{d\zeta} = \frac{\epsilon}{2\cos^2\phi\sin\phi} Re\left[\int_{-\infty}^{+\infty} d\xi \tilde{P}(v)\frac{\partial v^*}{\partial \zeta}\right], \qquad (29)$$

where the functions on the right-hand side of Eq. (29) must be also calculated in the adiabatic approximation using the solution (24). Equation (29) is the basis of the adiabatic approximation of the perturbation theory for dark solitons.

B. Varying background

If the perturbation $\epsilon P(u)$ in Eq. (21) does not vanish at $|x| \to \infty$, it will certainly affect the background wave. This is the standard case of dissipative perturbations which produce a slow decay of the background amplitude. Taking the limit $|x| \to \infty$ and being interested in the evolution of the nonpropagating background $u_b(z)$ itself (i.e., that which does not depend on x), we obtain the equation for Eq. (21) the background evolution,

$$i\frac{du_b}{dz} - |u_b|^2 u_b = \epsilon P(u_b) \; . \qquad (30)$$

Equation (30) allows us to find the law describing the background evolution in the presence of perturbations. Generally, a solution of Eq. (30), $u_b(z)$, may be written in the form

$$u_b(z) = u_0(z)e^{i\theta(z)} \; , \qquad (31)$$

where the function $u_0(z)$ characterizes the change of the background amplitude and $\theta(z)$ is a varying phase of the background wave. To describe now the evolution of a dark soliton on such a varying background we should remove the background by the transformation [cf. Eq. (22)]

$$u(z,x) = u_0(z)e^{i\theta(z)}v(x,z), \qquad (32)$$

and to find an effective nonlinear equation for the function $v(x,z)$. In many cases (e.g., in the cases analyzed below) such an equation may be transformed into a perturbed NLS equation (23) after a change of the variables $\xi = \xi(x,z)$, $\zeta = \zeta(x,z)$, so that this will allow us to apply immediately the result given by Eq. (29).

IV. APPLICATIONS

In this section we apply the general analytical approach presented above to several particular cases which correspond to physically important perturbations to optical dark solitons. As a matter of fact, two of the problems we analyze here have been mentioned earlier in the literature (the effects of the Raman self-induced scattering and linear absorption); however, we present these cases as well to show how all particular examples follow from our general formalism.

A. Gain with saturation

As is well known, in the problem of the propagation of spatial solitons nonlinearity is usually associated with two-photon absorption which, in fact, appears as a byproduct of enhanced nonlinearity [40,45]. In the presence of either two-photon absorption or gain, the stationary self-localized states of a light wave are no longer possible; but in the case of a combined effect, when the two-photon absorption is compensated by a gain, the stationary solution in the form of a fundamental dark soliton was shown to exist (see, e.g., [22]). However, as has been shown in [30], such a solution is stable to the action of symmetric perturbations but it becomes unstable to asymmetric perturbations. In this subsection, we address the question of stability of a dark soliton using the perturbation theory for dark solitons presented in Sec. III.

In the case of a saturated gain (or, equivalently, in the presence of a linear gain and two-photon absorption), the modified NLS equation for a self-defocusing nonlinear medium takes the form [22, 30]

$$i\frac{\partial u}{\partial z} + \frac{1}{2}\frac{\partial^2 u}{\partial x^2} - |u|^2 u = i\alpha u - iK|u|^2 u, \qquad (33)$$

where on the right-hand side the term $\propto \alpha$ represents the constant gain contribution and the term proportional to K accounts for the intensity-dependent saturation of the gain (e.g., due to the absorption). In the absence of any soliton, the background wave may be stabilized by a simultaneous action of the gain and absorption. Indeed, looking for the cw solution of Eq. (33) in the form $u(z) = u_0(z)\exp[i\theta(z)]$, we come to the following equations:

$$\frac{du_0}{dz} = u_0\left(\alpha - Ku_0^2\right), \quad \frac{d\theta}{dz} = -u_0^2, \qquad (34)$$

which show that a stable cw background with the amplitude

$$u_\infty^2 = \frac{\alpha}{K} \qquad (35)$$

may exist as a result of interplay between the effects of gain and nonlinear absorption. And what about a dark soliton on such a stable background? To analyze the dynamics of a dark soliton on the stable background with the amplitude (35), we use the main results of Sec. III A and look for a *nonstationary* solution of Eq. (33) in the form

$$u(z,x) = u_\infty v(z,x)\exp(-iu_\infty^2 z). \qquad (36)$$

It is easy to verify that the function $v(z,x)$ satisfies the equation

$$i\frac{\partial v}{\partial y} + \frac{1}{2}\frac{\partial^2 v}{\partial \tau^2} - (|v|^2 - 1)v = -iK(|v|^2 - 1)v, \qquad (37)$$

where y and τ are renormalized variables, $y = z(\alpha/K)$ and $\tau = x\sqrt{\alpha/K}$. Equation (37) clearly shows that, as soon as $|v| \neq 1$, the combined action of the gain and absorption produces an effective perturbation to the NLS equation which will certainly affect the dark-soliton propagation. To analyze the effect of the small perturbation

($\sim K$) from the left-hand side of Eq. (37) on a dark soliton, we use Eq. (29) from Sec. III A and obtain the following equation for the soliton phase angle:

$$\frac{d\phi}{dz} = \frac{\alpha}{3}\sin(2\phi). \tag{38}$$

Equation (38) is easily integrated to yield the result

$$\phi(z) = \tan^{-1}\left[\tan\phi(0)e^{\frac{2}{3}\alpha z}\right], \tag{39}$$

which clearly shows that only the soliton with $\phi(0) = 0$ is a stationary solution, whereas any deviation from this particular case will lead to the evolution of the soliton phase to reach a limit value, either $\phi(\infty) = \frac{\pi}{2}$ [for $\phi(0) > 0$] or $\phi(\infty) = -\frac{\pi}{2}$ [for $\phi(0) < 0$] corresponding to an infinite width and zero contrast of the soliton pulse. As follows from Eq. (39), any symmetric perturbation to a fundamental dark soliton which does not change effectively the condition $\phi(0) = 0$ to the soliton will not produce such an instability, whereas any asymmetric perturbation leads to a change of the phase angle and, subsequently, to the pulse decay reported in Ref. [30].

To compare the result (39) with direct numerical simulations, we present in Fig. 1 the function $\sin\phi(z)$ in the cases of three different initial conditions, $\phi(0) = 0, 0.15\pi$, and -0.15π, which clearly indicate that the stationary state with $\phi(0) = 0$ is unstable. The solid curves are the results given by Eq. (39) and, as follows from Fig. 1, they are in excellent agreement with the results of direct numerical simulations shown by diamond marks. Figure 2 shows the contour plots of the dark-soliton evolution in the presence of a gain with saturation as in Figs. 1(a), 1(b), and 1(c), respectively. The radiation observed in all the cases may be explained by the fact that the adiabatic solution, even at $\phi(0) = 0$, is not an exact one, and it differs from the exact dark soliton which has a more complicated phase function (see, e.g., Ref. [22]).

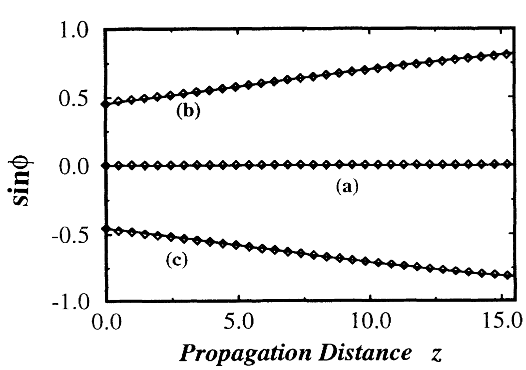

FIG. 1. Evolution of the soliton phase angle given by the function $\sin\phi(z)$, in the Kerr medium with gain at $\alpha = 0.1$ and two-photon absorption at $K = 0.1$ for three different initial values of the phase angle, (a) $\phi(0) = 0$, (b) $\phi(0) = 0.15\pi$, and (c) $\phi(0) = -0.15\pi$. The solid curves are from Eq. (39) and the diamond marks are the results of the numerical simulation of Eq. (33) with the stabilized background determined by Eq. (35).

B. Two-photon absorption

Recently, several types of optical switching devices have been proposed to be based on the propagation and interaction of spatial dark solitons [29]. In order to reduce the power for the soliton formation and the switching threshold, one has to use materials with higher nonlinearities than that of silica. Another way to improve nonresonant nonlinearities is to use the enhancement that occurs near two-photon resonances. However, in many cases an enhanced nonlinear coefficient is accompanied by an enhancement of the two-photon absorption (TPA) coefficient. This is the case, for example, for dark spatial solitons observed in the semiconductor ZnSe which is known to be an instantaneous, defocusing nonlinear medium with a rather strong TPA [11, 40]. Therefore, to use spatial solitons for all-optical switching, one should analyze the effect of TPA on the propagation of solitons. Such an analysis was presented recently by Silberberg [40] for the case of bright solitons. Here we analyze the effect of TPA on dark solitons. We consider the NLS equation modified as follows (see, e.g., Refs. [40, 39]):

$$i\frac{\partial u}{\partial z} + \frac{1}{2}\frac{\partial^2 u}{\partial x^2} - |u|^2 u = -iK|u|^2 u. \tag{40}$$

Here u is the normalized field amplitude, z and x are the normalized propagation and transverse coordinates, and K is the normalized TPA coefficient, $K = \beta/2k_0 n_2$, where k_0 is the free-space wave vector, and β and n_2 are the nonlinear absorption and refractive index coefficients, respectively.

In the absence of the TPA contribution, i.e., at $K = 0$, Eq. (40) describes the case of a defocusing Kerr nonlinearity where dark spatial solitons may propagate on a modulationally stable background wave $u = u_0 \exp(-iu_0^2 z)$, u_0 being the background amplitude. The nonlinear absorption, even when small, leads to attenuation of the cw background wave, and the wave's amplitude and phase become slowly dependent on $Ku_0^2(0)z$ according to

$$u_0(z) = \frac{u_0(0)}{\sqrt{1 + 2Ku_0^2(0)z}}, \tag{41}$$

$$\theta(z) = \int_0^z u_0^2(z')dz' = \frac{1}{2K}\ln\left[1 + 2Ku_0^2(0)z\right]. \tag{42}$$

To take into account explicitly the TPA-induced evolution of the background wave, we apply the following transformation:

$$u(z, x) = u_0(z)e^{i\theta(z)}v(z, x), \tag{43}$$

where $u_0(z)$ and $\theta(z)$ are changing according to Eqs. (41) and (42), and obtain the following equation for the function v:

$$i\frac{\partial v}{\partial \zeta} + \frac{1}{2}\frac{\partial^2 v}{\partial \xi^2} - (|v|^2 - 1)v = -iK(|v|^2 - 1)v, \tag{44}$$

where ζ and ξ are new coordinates which are connected with z and x by the following differential relations, $d\zeta =$

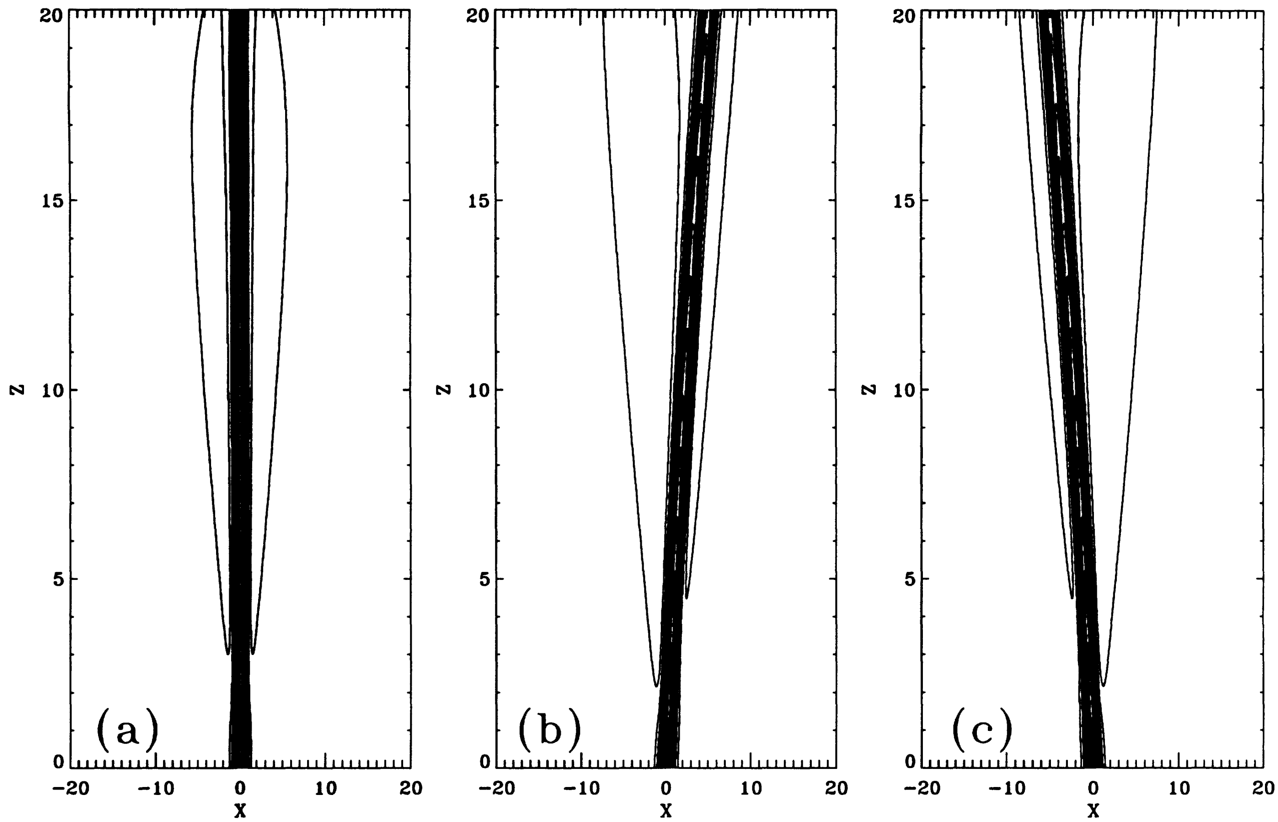

FIG. 2. Contour plots of numerical simulation showing the dark-soliton evolution in the presence of gain with saturation. The plots (a), (b), and (c) correspond to the curves (a), (b), and (c) in Fig. 1 for $\phi(0) = 0$, 0.15π, and -0.15π, respectively.

$u_0^2(z)dz$ and $d\xi = u_0(z)dx$. After such a transformation, the resulting Eq. (44) has *a vanishing perturbation* $\sim K$ and it may now be analyzed by means of the perturbation theory for solitons.

Equation (44) looks exactly like Eq. (37) of the preceding section. However, the difference is the new space-dependent variables ζ and ξ used in Eq. (44). As a result, the equation for the soliton phase angle ϕ in the primary variables takes the form [cf. Eq. (38)]

$$\frac{d\phi}{dz} = \frac{1}{3}Ku_0^2(z)\sin(2\phi), \tag{45}$$

where the background amplitude $u_0(z)$ decays according to Eq. (41). Equation (45) may be easily integrated to give the result [cf. Eq. (39)]

$$\phi(z) = \tan^{-1}\left\{\tan\phi(0)\left[1 + 2Ku_0^2(0)z\right]^{1/3}\right\}. \tag{46}$$

One of the main characteristics of the dark-soliton switching devices is the so-called *steering angle* [29]. It is easy to see that the total shift of the dark soliton along the x axis is given by the relation $\int^z dz' u_0(z')\sin\phi(z')$, so that the steering angle χ may be defined through the local transverse velocity,

$$W(z) = \tan\chi = u_0(z)\sin\phi(z). \tag{47}$$

The important conclusion based on Eq. (47) is the following: When the dark-soliton propagates in the presence of TPA on a decaying background $u_0(z)$, the function $\sin\phi(z)$ grows slowly keeping, at least for small $\phi(0)$, the product (47) almost unchanged. This simply means that the steering angles for the switching devices based on dark-soliton propagation are almost preserved in a Kerr nonlinear medium in the presence of TPA. From the physical point of view, this important property simply follows from the nature of the nonlinear absorption: the background intensity decays faster than the central minimum in the soliton.

In order to confirm our analysis, we have checked the results (41), (46) of the adiabatic approximation by comparing them with numerical solutions of Eq. (40). Figures 3(a) and 3(b) show the evolution of the background $u_0(z)$, the soliton phase angle defined by $\sin\phi(z)$, and the transverse soliton velocity $W(z) = u_0(z)\sin\phi(z)$ for two different values of $\phi(0)$. The analytical results (solid curves) based on Eqs. (41) and (46) are in a perfect agreement with the results of the numerical simulations (open diamonds) and, as may be seen, the steering angle is almost preserved provided $\phi(0)$ is small. Small deviations of the numerical data from the adiabatic relationship are caused by a transition radiation which slightly changes the intensity of the background (see Fig. 4).

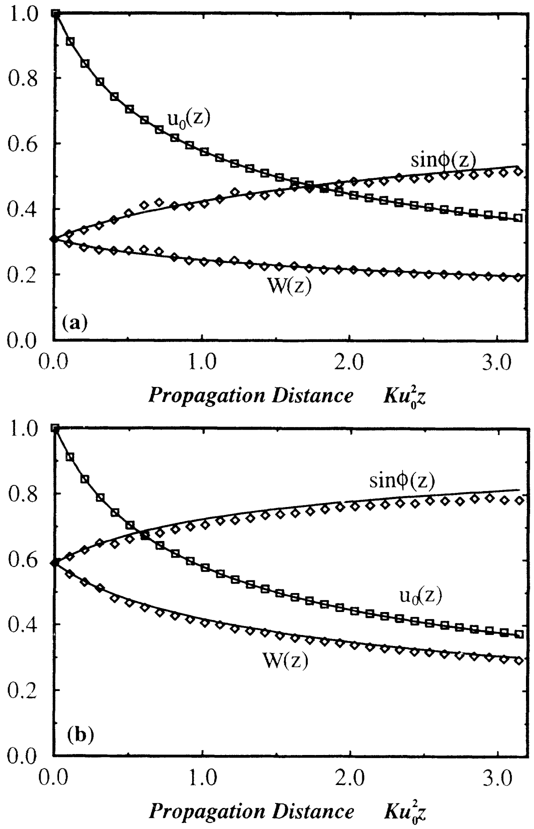

FIG. 3. Evolution of the background amplitude, $u_0(z)$, the normalized phase angle, $\sin\phi(z)$, and the transverse velocity of the dark soliton, $W(z)$, along the Kerr medium with TPA at $K = 0.05$. The solid curves are from Eqs. (41) and (46), the diamond marks are obtained from numerical simulations to Eq. (40) at (a) $\phi(0) = 0.1\pi$ and (b) $\phi(0) = 0.2\pi$.

C. Linear absorption

It is important to compare the result (46) with the corresponding result for the linear absorption described by the contribution $\epsilon P(u) = -i\gamma u$ on the right-hand side of Eq. (40) instead of the term $-iK|u|^2u$. As follows from the corresponding perturbed NLS equation, in this case the background wave decays according to the exponential law,

$$u_0(z) = u_0(0)\, e^{-\gamma z},$$
$$\theta(z) = \int_0^z u_0^2(z')\, dz' = \frac{u_0^2(0)}{2\gamma}(1 - e^{-2\gamma z}). \tag{48}$$

As in the case of the TPA dynamics, first of all we remove the background evolution by the transformation (43) where this time the functions $u_0(z)$ and $\theta(z)$ are given by Eq. (48). The important result of such a transformation is that the effective equation (44) for the function $v(\zeta, \xi)$ is the NLS equation *without* perturbations. This immedately implies that the transformation (43) does allow the exclusion of the effect of the linear absorption considering the pulse evolution in the new reference frame, so that the soliton phase angle does not change,

$$\frac{d\phi}{dz} = 0. \tag{49}$$

This result does explain why the final equation of Ref. [24] was correct in spite of the fact that the integrals, formally written there, are divergent: In Eqs. (7) and (8) of Ref. [24] two infinities arising from the integrals calculated for the dark soliton on an infinite background finally cancel each other. However, as may be easily

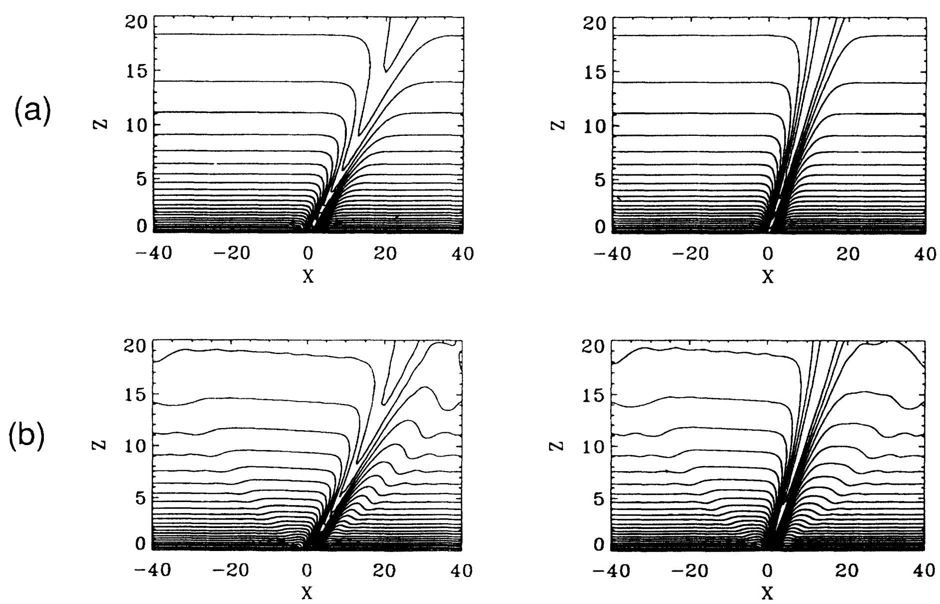

FIG. 4. Contour plots demonstrating the effect of TPA on a dark soliton for $K = 0.05$ and $\phi(0) = 0.2\pi$ (left column) and $\phi(0) = 0.1\pi$ (right column) based on (a) adiabatic approximation given by Eqs. (41) and (46) and (b) numerical simulations of Eq. (40).

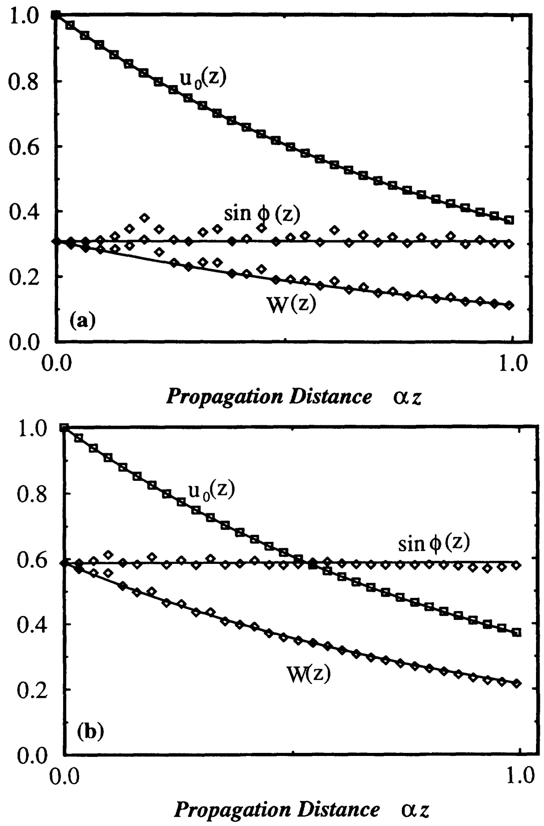

FIG. 5. The same as in Figs. 3(a) and 3(b) but for the case of the linear absorption at $\gamma = 0.1$; (a) $\phi(0) = 0.1\pi$ and (b) $\phi(0) = 0.2\pi$.

checked, any effort to apply the formulas (7) and (8) of Ref. [24] to analyze the effect of another perturbation (e.g., a nonlinear absorption) will not be successful because, as we have shown above, the dynamics of the perturbation-induced dark soliton does include the evolution of the soliton phase angle as well, which is trivial only for the case of a linear absorption.

Figures 5 and 6 (which are similar to Figs. 3 and 4 for the case of the two-photon absorption) show the comparison of analytical and numerical results at $\gamma = 0.1$ and two initial values, $\phi(0) = 0.1\pi$ and $\phi(0) = 0.2\pi$. Again, we observe a rather good agreement of the numerical results with the adiabatic approximation and small deviations observed are mainly caused by radiation effects.

D. Raman self-frequency shift of dark solitons

As has been mentioned above, the perturbation theory for dark solitons elaborated in the present paper may be effectively applied to analyze the soliton propagation in optical fibers, i.e., temporal dark solitons [6–8]. In such a case, the different nature of these two physical problems gives rise to different physically important perturbations to dark solitons. When the pulse duration in fibers reaches the subpicosecond regime, it becomes necessary to include higher-order dispersion effects. These effects are presented by higher-order derivatives in the effective NLS equation for the wave envelope [46]. Stimulated Raman scattering (SRS) is known to be one of the dominant effects for very short optical pulses. For bright solitons this effect causes the so-called self-frequency shift [47, 48], whereas for dark solitons, the self-frequency shift at the initial stage of the pulse evolution [14] leads finally to a decay of dark solitons [18, 23, 33]. From the physical point of view, the SRS effect originates from the noninstanteneous, delayed response of the fiber nonlinearity. This effect may be described in the temporal domain by a response function that has a form of a decaying sinusoidal oscillations [49]. The Raman contribution to the nonlinear refractive index may be taken into account in a rather general form,

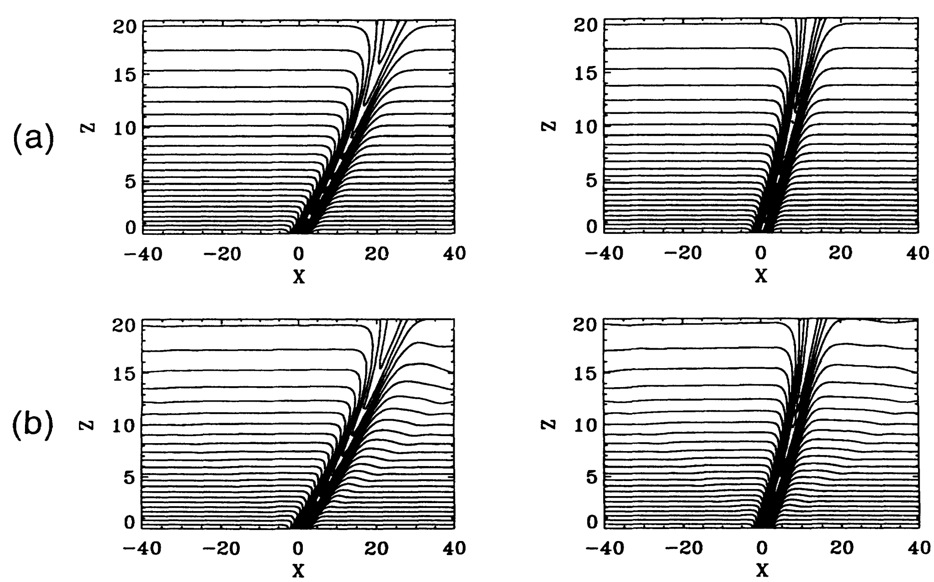

FIG. 6. The same as in Figs. 4(a) and 4(b) but for the case of the linear absorption at $\gamma = 0.1$.

$$n_2|u|^2 \to n_2\left[(1-\alpha)|u|^2 + \alpha\int_{-\infty}^{t} dt'|u(t')|^2 f(t-t')\right], \tag{50}$$

where α is the fraction of the total (low-frequency) nonlinearity with a delayed response, and $f(t)$ is the Raman response function (see, e.g., Ref. [49]). The Raman response function of fused silica is extremely short, so that Eq. (50) may be considered in the local approximation expanding the function $|u(t-s)|^2$ in the integrand of Eq. (50) (here $s = t - t'$) in the Taylor series around t to obtain the contribution into the NLS equation (21) in a local form,

$$\epsilon P(u) = \epsilon u \frac{\partial}{\partial t}(|u|^2), \tag{51}$$

ϵ being proportional to the Raman gain parameter α. In such a form, the effect of SRS may be analyzed as a perturbation to the standard NLS dynamics and for dark solitons it was investigated numerically [14, 23, 25] and analytically [18, 33]. As a matter of fact, the general formula describing the dark-soliton propagation in the presence of SRS was obtained in [33], whereas its small-amplitude limit was derived earlier [18] using the so-called asymptotic approach which in the main order gives a perturbed Korteveg–de Vries–Burgers equation. In the present section we consider the effect of SRS to dark solitons once more, trying (i) to include into the analytical approach the response function of a rather general form, i.e., to take into account the nonlocal term from Eq. (50), and (ii) to use the SRS perturbation as a particular example for our general perturbation theory presented in Sec. III. Additionally, as will follow from our analysis, taking the soliton phase angle as a parameter, it is possible to describe the SRS effect to a dark soliton using only one equation (in Refs. [18, 33] two equations are responsible for such a dynamic, the equations being different for the opposite directions of the soliton propagation).

We start our analysis considering the perturbed NLS equation

$$\frac{\partial u}{\partial z} - \frac{1}{2}\frac{\partial^2 u}{\partial t^2} + |u|^2 u = -\tilde{\epsilon} u \int_{-\infty}^{t} |u(t')|^2 G(t-t')dt'. \tag{52}$$

To write Eq. (52), we assume that the first term of the expansion of the nonlocal part of Eq. (50),

$$u(t)\int ds f(s)|u(t-s)|^2 \to |u|^2 u \int_{-\infty}^{+\infty} f(s)ds - u\frac{\partial}{\partial t}(|u|^2)\int_{-\infty}^{+\infty} s f(s)ds, \tag{53}$$

is already included into the main Kerr-type nonlinearity $|u|^2 u$ so that the "renormalized" response function $G(s)$ has the property $\int_{-\infty}^{+\infty} G(s)ds = 0$. In fact, such a property corresponds to a constant background $u = u_0 \exp(iu_0^2 z)$. Considering the response function as very short, one may expand the integral in Eq. (52) recovering Eq. (51) with

$$\epsilon \equiv \tilde{\epsilon}\int_{-\infty}^{+\infty} sG(s)ds. \tag{54}$$

If the parameter $\tilde{\epsilon}$ max$\{G\}$ is small, we may treat the right-hand side of Eq. (52) as a perturbation to a dark soliton applying the main results of Sec. III [note the change of the sign in Eq. (52) corresponding to the temporal domain]. Applying the perturbation theory to the dark soliton (24) [with the change $z \to -z$ because of the difference in the signs for Eqs. (21) and (52)], and changing the order of integration, one may come to the following equation for the soliton phase angle:

$$\frac{d\phi}{dz} = \tilde{\epsilon} u_0 \cos\phi \int_{-\infty}^{+\infty} ds\, G\left(\frac{s}{u_0\cos\phi}\right) \times \int_{-\infty}^{+\infty} d\tau \frac{\tanh\tau}{\cosh^2\tau\,\cosh^2(\tau-s)}. \tag{55}$$

The integral over τ may be easily calculated using the standard formula to simplify the expression in the integrand. As a result, we obtain the following equation:

$$\frac{d\phi}{dz} = \tilde{\epsilon} u_0 \cos\phi \int_{-\infty}^{+\infty} dsG\left(\frac{s}{u_0\cos\phi}\right) F(s), \tag{56}$$

where

$$F(s) = \frac{2}{\sinh^2 s \tanh^2 s}\left[s(3-\tanh^2 s) - 3\tanh s\right]. \tag{57}$$

As follows from Eq. (57), the local approximation may be used only in the case when the Raman response function is short in comparison with the soliton width $\sim (u_0\cos\phi)^{-1}$. In this case we may simply expand the function $F(s)$ into the Fourier series as $F(s) \simeq F(0) + sF'(0)$ and obtain the resulting equation in the form of a perturbed NLS equation as an expansion in the inverse width of the soliton $\sim (u_0\cos\phi)$, the latter is assumed to be small in comparison with the parameter T^{-1}, T being the characteristic decay of the response function,

$$\frac{d\phi}{dz} = \frac{4}{15}\epsilon u_0^3 \cos^3\phi, \tag{58}$$

where

$$\epsilon \equiv \tilde{\epsilon}\int_{-\infty}^{+\infty} y\, G(y)dy. \tag{59}$$

Equation (58) follows, as a matter of fact, from the result of Ref. [33] after a simple change of variables, and in the small amplitude limit, i.e., when ϕ is close to $\pi/2$, it coincides with the result of Ref. [18]. However, the result (56), (57) is much more general in the sense that it does allow to take into account effects of various *nonlocal* perturbations to the dynamics of dark solitons.

To support our analytical predictions numerically, we have carried out numerical simulations taking, for simplicity, the NLS equation with a local contribution (51) due to the SRS effect. The results are presented in Figs. 7 and 8. Comparison of the numerical simulation results

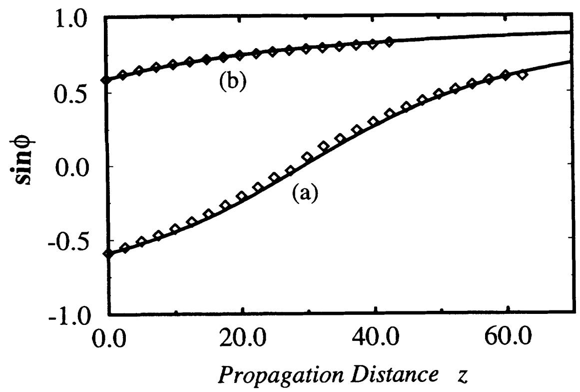

FIG. 7. The evolution of the phase angle of the dark soliton in the presence of the Raman self-scattering effect described by Eqs. (21) and (51) at $\epsilon = 0.1$; (a) $\phi(0) = -0.2\pi$ and (b) $\phi(0) = 0.2\pi$.

with Eq. (58) is presented in Fig. 7 as the soliton phase angle given by the function $\sin\phi$ vs the propagation distance at two different initial values, $\phi(0) = -0.2\pi$ and $\phi(0) = +0.2\pi$. Figure 8 shows the evolution of a dark soliton with an initial negative [Fig. 8(a)] or positive [Fig. 8(b)] velocity in the presence of the SRS contribution at $\epsilon = 0.1$. As may be seen from those figures, Eq. (58) describes rather well the soliton dynamics which, as a matter of fact, corresponds to the transformation of a dark soliton with different initial values $\phi(0)$ into a small-amplitude dark soliton and its subsequent decay due to the continuous SRS-induced frequency and position shift.

To conclude the present section, we would like to mention one more important effect which might become important for temporal dark solitons. This effect is a contribution of the higher-order dispersion, the latter becomes important for rather narrow optical pulses. The third-order dispersion has been shown to produce a nontrivial dynamics of bright solitons (see, e.g., Refs. [50, 51]). As for dark solitons, a simple analysis based on the perturbation theory displays a stability of dark solitons in the framework of the adiabatic approximation. That is definitely true for a rather small contribution of the third-order dispersion into the effective NLS equation. However, in the case when the second- and third-order dispersion contributions become of the same order (e.g., in the vicinity of the so-called zero-dispersion point) dark solitons may still exist but with sufficiently changed properties (see, e.g., [21]).

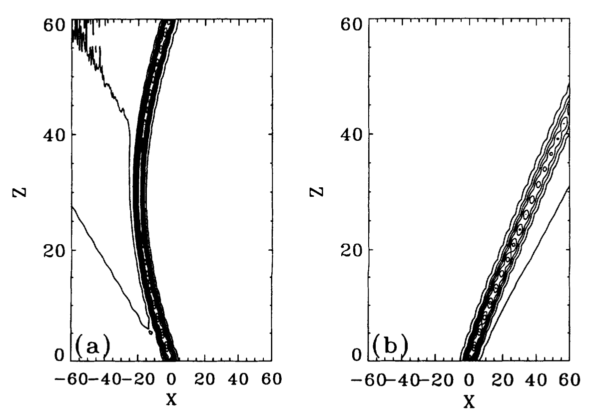

FIG. 8. Contour plots of the dark-soliton propagation corresponding to Fig. 7; (a) $\phi(0) = -0.2\pi$ and (b) $\phi(0) = 0.2\pi$.

V. DARK SOLITONS ON A BACKGROUND OF FINITE EXTENT

In standard experiments involving the dark-soliton propagation, dark pulses are created on a background (carrier) pulse of a finite duration, and this pulse usually has the shape of a long bright pulse [6–8]. This is also the case of spatial dark solitons observed as regions of a decreased intensity in a beam of a finite width [9, 11]. Therefore, the interpretation of the experimental results as the soliton propagation could be questionable because the background pulse, being only several times longer than the dark pulses observed, spreads significantly and develops a *frequency chirp*; such a spreading is well known in the theory of linear waves as a dispersive spreading (temporal domain) or difraction (spatial domain). Tomlinson *et al.* [15] demonstrated by means of direct numerical simulations that optical dark pulses superimposed upon backgrounds only 10 times wider than the soliton width can exhibit stable solitonlike propagation for relatively short distances. During propagation, the background pulse spreads, reduces its intensity, and develops a frequency chirp but, nevertheless, dark pulses created on such a finite-width background did not display a drastic change and they adiabatically maintained their soliton characteristics. As has been pointed out by Gredeskul *et al.* [19], for the finite-width background the corresponding eigenvalue problem of the inverse scattering transform (IST) has no eigenvalues of the discrete spectrum and dark pulses created on a vanishing background correspond instead to the so-called *quasistationary* states of the IST eigenvalue problem. This simply means that these dark pulses are not proper solitons and they disappear as soon as the propagation distances are taken to be larger. All these results, supported by direct numerical simulations [15], are in a good agreement with experimental investigations of dark solitons. However, those results do not give a clear physical explanation why the dark pulses, even being not proper solitons, do not change significantly when the background itself spreads, reduces its intensity, and develops a frequency chirp. In the present section we apply the perturbation theory developed above to explain analytically the phenomena observed at the dark-soliton propagation on backgrounds of finite extent.

In fact, the problem of a dark soliton on a finite-width background is not that formally considered as a perturbative problem. However, as we show below, it may be reduced to that problem considering first the background evolution.

Let us consider the NLS equation *without* perturbations,

$$i\frac{\partial u}{\partial z}+\frac{1}{2}\frac{\partial^2 u}{\partial x^2}-|u|^2u=0. \tag{60}$$

For vanishing boundary conditions, Eq. (60) is known to describe the spreading pulses (or beams) which undergo enhanced broadening and chirping. Let us take such a quasilinear solution in a rather general form,

$$u(x,z)=u_0(x,z)e^{i\theta(x,z)}, \tag{61}$$

where we have introduced the pulse amplitude $u_0(x,z)$, and phase $\theta(x,z)$. As is well known, Eq. (60) is exactly integrable and it may be analyzed by means of the IST technique [44]. For vanishing boundary conditions Eq. (60) discribes an asymptotic decay of the so-called *nonsolitonic pulses*, and the characteristics of such a dispersively spreading pulse may be found with the help of the IST technique [46],

$$u_0^2(x,z)\simeq\frac{1}{4\pi z}\ln\left|a\left(-\frac{x}{2z}\right)\right|^2, \quad z\gg 1, \tag{62}$$

where $a(\lambda)$ is the so-called Jost coefficient introduced in the IST method applied to Eq. (60) with the boundary conditions $u\to 0$ at $|x|\to\infty$. The evolution of the pulse phase $\theta(x,z)$ displays, in its turn, an enhanced chirping.

Let us consider now the evolution of a dark pulse superimposed upon such a spreading background, looking for a solution of Eq. (60) in the form

$$u(x,z)=u_0(x,z)e^{i\theta(x,z)}v(x,z), \tag{63}$$

where $v(x,z)$ falls off fast as x increases. Substituting Eq. (63) into Eq. (60) and assuming that the background function $u_0(x,z)\exp[i\theta(x,z)]$ is a solution of the NLS equation (60), we obtain the following equation for the function $v(x,z)$,

$$i\frac{\partial v}{\partial z}+\frac{1}{2}\frac{\partial^2 v}{\partial x^2}-u_0^2(|v|^2-1)v=-2\frac{\partial}{\partial x}\left[\ln(u_0e^{i\theta})\right]\frac{\partial v}{\partial x}, \tag{64}$$

where $u_0\equiv u_0(x,z)$ and $\theta\equiv\theta(x,z)$ are varying amplitude and phase, respectively. Using the new (approximate) variables introduced for slowly varying $u_0(x,z)$ according to the relations $d\zeta\approx u_0^2dz$ and $d\xi\approx u_0dx$, we come to the following equation:

$$i\frac{\partial v}{\partial \zeta}+\frac{1}{2}\frac{\partial^2 v}{\partial \xi^2}-(|v|^2-1)v=-2\frac{\partial}{\partial \xi}\left[\ln(u_0e^{i\theta})\right]\frac{\partial v}{\partial \xi}, \tag{65}$$

which may be treated as a perturbed NLS equation (23), allowing us to apply the results of the perturbation theory for dark solitons described in Sec. III.

If we take the pulse $v(\zeta,\xi)$ as a dark soliton (24), the pulse evolution under the action of the right-hand side of Eq. (65) may be analyzed with the help of Eq. (29) to obtain the equation for the soliton phase angle,

$$\frac{d\phi}{d\zeta}=\frac{1}{2}\cos^2\phi\int_{-\infty}^{+\infty}\frac{dZ}{\cosh^2 Z}\left(\frac{1}{u_0}\frac{\partial u_0}{\partial Z}\right), \tag{66}$$

where $u_0=u_0(Z,\zeta)$ is considered in the reference frame moving with the soliton,

$$Z=\cos\phi(\zeta)\left[\xi-\int d\zeta'\sin\phi(\zeta')\right].$$

Equation (66) is valid for a rather arbitrary background pulse, and it clearly shows that the evolution of a dark soliton *does not depend on variations of the background phase*, so that the enhanced frequency chirp developed by the background does not affect the dark pulse, and the pulse maintains adiabatically its properties. Additionally, as follows from Eq. (66), the change of the soliton phase angle depends not on the extension and intensity of the background but rather on the background's slope at the point where the soliton is. In practice, this result simply demonstrates that properly selecting the input background shape we may keep the steering angle of a dark soliton, $W=u_0\sin\phi$, almost unchanged because the background decay will be compensated by the interal dynamics of the soliton. Unfortunately, we cannot obtain the final analytical result from Eq. (66) because the evolution of the background $u_0(x,z)$ is known only in less interesting asymptotic region $z\gg 1$, see Eq. (62). However, taking into account the analytical expression (66), it is easy to carry out numerical simulations to observe the phenomenon predicted. First of all, in the numerical simulations the effect of the background frequency chirp is not observed to have any influence on the soliton propagation, this result is known since the work of Tomlinson

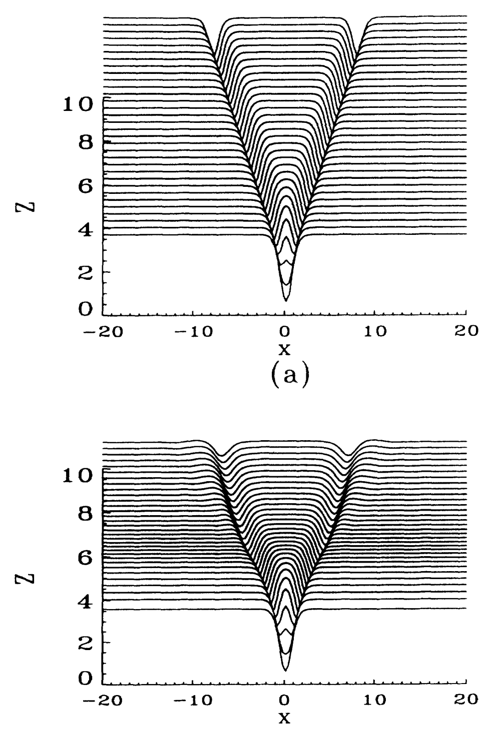

FIG. 9. Creation of a pair of dark solitons on an infinite background with an even initial condition: (a) constant-amplitude background, and (b) the same conditions as in (a), but the background amplitude decreases according to the maximum amplitude of a dispersively spreading Gaussian pulse shown in Fig. 10(a).

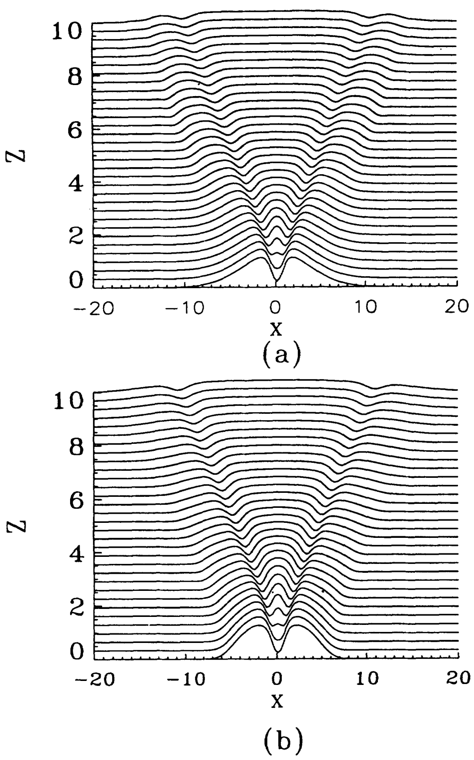

FIG. 10. Dark-soliton propagation on a finite-width background: (a) Gaussian input background pulse, and (b) super-Gaussian input background pulse.

et al. [15]. Second, the dynamics of the soliton phase angle vs the intensity of the constant or decaying background may be clearly seen in Figs. 9(a) and 9(b) and 10(a) and 10(b). In Fig. 9(a) we show a typical evolution of an even dark pulse on a constant background when the input pulse, having the same phases at the pulse edges, splits into a pair of two (symmetric) dark solitons propagating at (opposite in sign) constant velocities, and this corresponds to a constant steering angle between the solitons. If we consider the infinite background whose intensity asymptoticaly vanishes according to the amplitude of a dispersively spreading bright pulse, the transverse soliton velocity is clearly seen to be varying, and it is directly proportional to the background intensity [see Fig. 9(b)]. This effect may be compensated, however, by the evolution of the phase angle which depends, according to Eq. (66), on the slope of the finite-width background. Figure 10(a) shows that the steering angle is almost preserved for the Gaussian input background pulse, whereas increasing the background slope leads to the effect opposite to that shown in Fig. 9(b). This is demonstrated in Fig. 10(b) for the so-called super-Gaussian input background pulse.

Thus, the general result (66) does allow us to understand the main features of the dark-soliton dynamics on the backgrounds of finite extent. First, the evolution of a dark soliton does not depend on an enhanced frequency chirp developed by the spreading background. Second, for a decaying background the phase angle of a dark soliton becomes changed according to Eq. (66), and this change is opposite to the effect produced by the vanishing background intensity. As a result, for the Gaussian input background the transverse velocity of the dark pulse is almost preserved by recovering the property observed for the constant-amplitude infinite background [cf. Figs. 9(a) and 10(a)].

VI. CONCLUSIONS

In conclusion, we have proposed a simple analytical approach to describe the perturbation-induced dynamics of dark solitons. This approach allows us to treat analytically the cases of constant as well as varying background. We have applied our general formalism to describe the effect of several physically important perturbations to optical dark solitons, and we have compared the results of the adiabatic approximation to those of the corresponding numerical simulation. This comparison has displayed a rather good agreement, and it has supported our main conclusion that the adiabatic approach does allow us to describe with high accuracy the dark-soliton evolution provided the dynamics of the slowly varying background wave is treated in a self-consistent manner. We have also pointed out that the analogous method is useful to explain the pulse propagation on a finite-width background, and in this case, as follows from our analysis, the frequency chirp of the decaying and dispersively speading background does not give a contribution to the pulse dynamics. Finally, we have summarized in Table I all the particular cases of the physically important perturbations to optical dark solitons considered in the present paper, and we have shown separately how such perturbations change the background amplitude and the soliton phase angle, respectively. As follows from Table I, in most of the cases analyzed here the perturbation-induced dynamics of a dark soliton may be understood as a combined effect of the background evolution and the change of the soliton phase angle.

TABLE I. Effect of different perturbations to the dark-soliton dynamics.

Type of perturbation	Background	Soliton phase angle
Two-photon absorption	Varying	Varying
Linear absorption	Varying	Constant
Raman scattering	Constant	Varying
Gain with saturation	Constant	Varying
Finite-width background	Varying	Varying
Higher-order dispersion	Constant	Constant

The general approach developed here as well as a rather general physics of dark solitons which may be observed in nonlinear models of very different physical origin, allow us to believe that our method and the results will be useful for other physical systems supporting propagation of (temporal or spatial) dark solitons.

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